REALIZATION OF ABSTRACT CONVEX GEOMETRIES BY POINT CONFIGURATIONS

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A closure space $(J, -)$ is called *a convex geometry* (see, for example, [1]), if it satisfies *the anti-exchange axiom*, i.e.

$$x \in \overline{A \cup \{y\}} \text{ and } x \notin A \text{ imply that } y \notin \overline{A \cup \{x\}}$$
$$\text{for all } x \neq y \text{ in } J \text{ and all closed } A \subseteq J.$$

Given a closure space, one can associate with it the lattice of closed sets $\mathrm{Cl}(J, -)$. It is well known that the lattice of closed sets of a *finite* convex geometry is join-semidistributive. The latter property is defined by

$$(\forall x, y, z \in L) \quad (x \vee y = x \vee z \quad \Rightarrow \quad x \vee y = x \vee (y \wedge z))$$

The following classical example of convex geometries shows how they earned their name. Given a set of points X in Euclidean space $\mathbb{R}^n$, one defines a closure operator on X as follows: for any $Y \subseteq X$, $\overline{Y} =$ *convex hull*$(Y) \cap X$. One easily verifies that such an operator satisfies the anti-exchange axiom. Thus, $(X, -)$ is a convex geometry. Denote by $\mathbf{C}o(\mathbb{R}^n, X)$ the closure lattice of this closure space, namely, the lattice of convex sets relative to X.

The current work was motivated by the following problem raised in [1]: *which lattices can be embedded into $\mathbf{C}o(\mathbb{R}^n, X)$ for some $n \in \omega$ and some finite $X \subseteq \mathbb{R}^n$? Is this the class of all finite join-semidistributive lattices?*

On the way to answer the above questions, one can address the associated problem raised in [2], and known as the

Edelman – Jamison Problem : *Characterize those finite convex geometries that are realizable as $\mathbf{C}o(\mathbb{R}^n, X)$.*

In the current paper we restrict ourselves to the case of $n = 2$ and point configurations in *general position*, i.e. where no 3 different points belong to one line. We formulate the hypothesis that a finite convex geometry is realizable by a point configuration on a plane, if two properties of very lucid geometrical nature hold: the so-called *splitting rule* and the *carousel rule.*

In one of major results of the paper we prove the hypothesis for all point configurations that have at most 2 points inside the n-gon. This extends the description of $\mathbf{C}o(\mathbb{R}^2, X)$ for the point configurations X that have one point inside a n-gon, given in [3]. We also confirm the hypothesis for all 6-point configurations on the plane.

In another part of our paper we discuss the connection between the Edelman-Jamison Problem and the Order Type Problem.

Following [4], call $t : J[3] \to \{1, -1\}$ an *order type on J*, if there is a function $f : J \to \mathbb{R}^2$ such that for all (a, b, c) in $J[3]$ one has

$$t(a, b, c) = \operatorname{sign}(f(a), f(b), f(c))$$

The point configuration $X = f(J)$ is then said to *realize* the order type t. In brief, t is an order type, if it represents the orientation of triples of some suitable point configuration.

The Order Type Problem : *Given any function* $t : J[3] \to \{1, -1\}$*, recognize whether it is an order type and, if it is, find some realizing point configuration.*

It is known that the Order Type Problem is NP-hard: that follows from the famous Mnëv's Universality Theorem [5]. We investigate whether the Order Type Problem can be polynomially reduced to Edelman-Jamison Problem.

It turns out that each order-type t generates a unique convex geometry $C(t)$, associated with its point realization. On the other hand, a realizable convex geometry C may have many realizations whose order-types are non-equivalent. The set of such non-equivalent order-types is denoted *Order-Types*(C).

We build a series of examples of convex geometries L_p to demonstrate the following:

Corollary 0.1. *The growth of* $|$*Order-Types*$(L_p)|$ *of convex geometries* L_p *of size* $\mathcal{O}(p)$ *cannot be* p*-polynomially bounded.*

This does not allow to straightforwardly reduce the Order-type Problem to Edelman-Jamison Problem.

On the positive side, we introduce the natural notion of a *simple* convex geometry, for which we can prove:

Theorem 0.2. *Given natural number* l*, let* $\mathcal{C}(l)$ *be the class of all finite simple convex geometries of depth* $\leq l$*, and let* $\mathcal{J}(l)$ *be the class of all candidate order-types whose unique associated convex geometry is in* $C(l)$*. Then the polynomial time decidability of the realizability of* $t \in \mathcal{J}(l)$ *is equivalent to the polynomial time decidability of the realizability of* $(J, -) \in C(l)$*.*

Here, the *depth* of the convex geometry indicates the number of its *layers*. The first *outside* layer L_1 of geometry $C = (J, -)$ is just a set of its *extreme* points. Considering the restriction of closure operator to $J \backslash L_1$, one obtains a convex geometry C_1, whose set of extreme points is the second layer L_2 of C etc.

Two points x, y of the same layer are are called *equivalent*, if $z \in \overline{\{x, u, v\}}$ iff $z \in \overline{\{y, u, v\}}$, for any u, v from the same layer. The geometry is called *simple*, if all its layers, except innermost, do not have equivalent points.

A candidate order type is a function $t : J[3] \to \{-1, 1\}$ for which an associated convex geometry can be defined. We show that it can be decided in polynomial time whether a given function t is a candidate order type. Besides, it can be checked in polynomial time, whether a given convex geometry is simple.

References

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